A Nearly Quadratic Improvement for Memory Reallocation

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SPAA 2024

The Memory Reallocation Problem

- **Input**: a sequence of item inserts and deletes.
- **Output**: on each item update, arrange the items to occupy non-overlapping regions of memory.
- **Goal** (intuitively) "minimize the total size of items moved".

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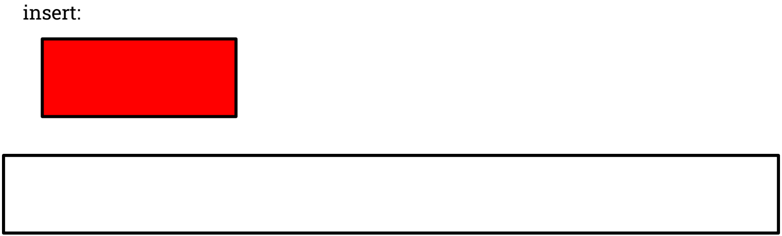

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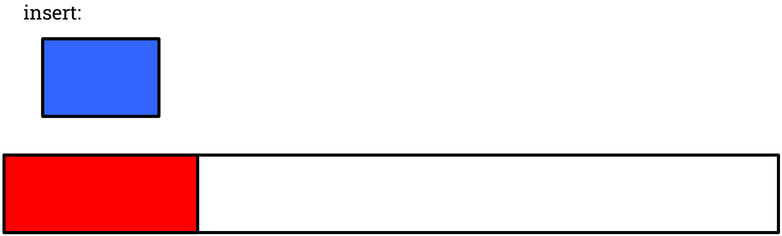

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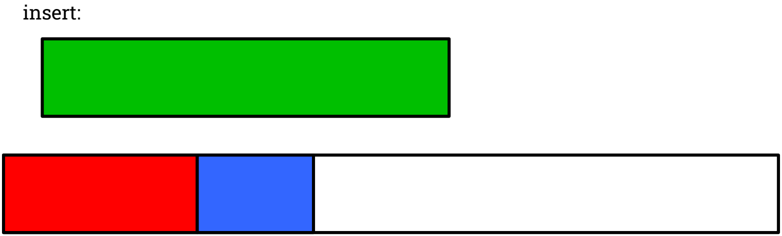

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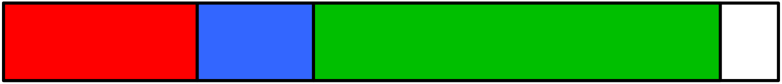

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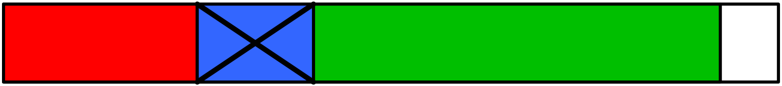

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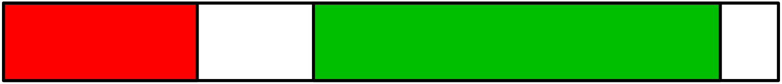

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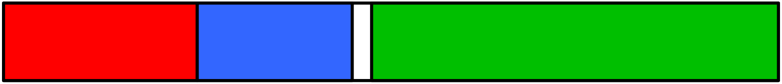

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Goal: Minimize update cost while handling load factor $1 - \varepsilon$.

Background

Proposition (Folklore Algorithm)

There is an allocator with update cost $O(\varepsilon^{-1})$.

Proof.

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- Re-arrange this interval and place the inserted item in it.

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There is an allocator with update cost $O(\varepsilon^{-1})$.

Theorem (Kuszmaul FOCS'23)

If all items have size at most $O(\varepsilon^4)$ then there is an allocator with expected update cost $O(\log \varepsilon^{-1})$.

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$\Omega(\varepsilon^{-1})$ expected update cost is required for items of size $\Theta(\varepsilon)$.

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Main Result

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Up next:
Prove a simpler version of this theorem to illustrate some ideas.

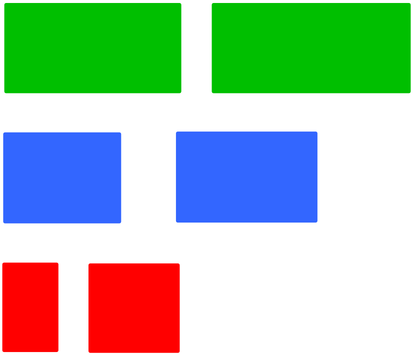
Theorem

There is an allocator for items with sizes in $[\varepsilon, 2\varepsilon]$ with average update cost $\widetilde{O}(\varepsilon^{-2/3})$.

Allocator Description

Partition $[\varepsilon, 2\varepsilon)$ into $\lceil \varepsilon^{-1/3} \rceil$ **size classes**.
i-th size class:

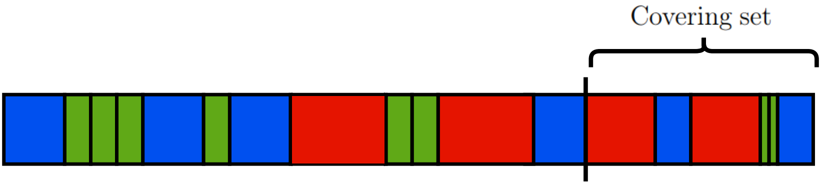
$$[\varepsilon + (i-1)\varepsilon^{4/3}, \varepsilon + i\varepsilon^{4/3}).$$



Allocator Description

Covering Set: Suffix of memory.
Let x_i = number of items of size class i.

Every $\lfloor \varepsilon^{-1/3} \rfloor$ updates the allocator performs an expensive **rebuild operation** where it rearranges all of memory to place the smallest $\min(x_i, \lfloor \varepsilon^{-1/3} \rfloor)$ items of size class i in the covering set.

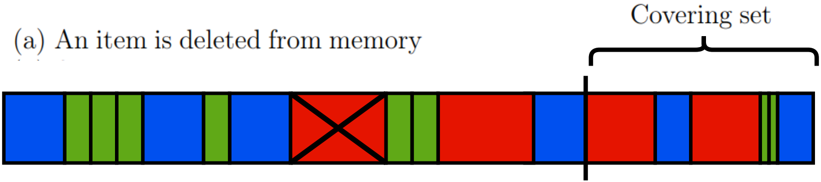


Allocator Description

Item Inserts: Add inserted items to the covering set. Place them after the final item in memory.

Allocator Description

Item Deletes:

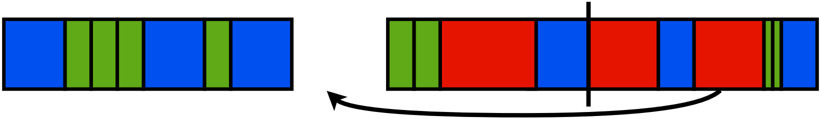


(a) An item is deleted from memory

Allocator Description

Item Deletes:

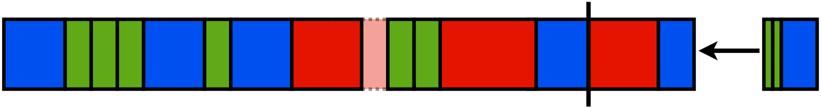
(b) SIMPLE replaces the deleted item with a smaller item of the same size class from the covering set



Allocator Description

Item Deletes:

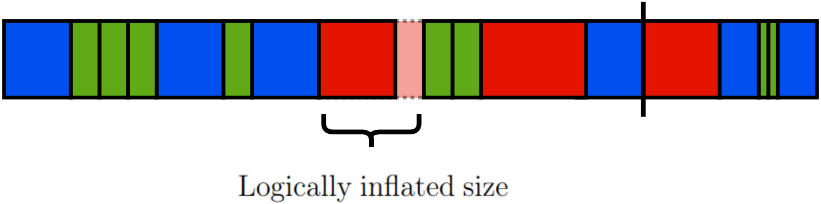
(c) SIMPLE compacts the covering set



Allocator Description

Item Deletes:

(d) The moved item is logically treated as being the size of the one it replaced until SIMPLE performs a full rebuild



Allocator Correctness

Lemma

The allocator is well defined and produces a valid allocation.

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Proof.

Periodic rebuilds prevent gaps from building up too much: we introduce up to $\varepsilon^{4/3}$ gap per delete, and rebuild after $\lfloor \varepsilon^{-1/3} \rfloor$ updates. $\square$

Allocator Performance

Lemma

The allocator achieves amortized update cost $O(\varepsilon^{-2/3})$.

Proof.

- The covering set consists of at most $O(\varepsilon^{-2/3})$ items, and so has total size at most $O(\varepsilon^{1/3})$.
- Compacting the covering set on each delete thus costs $O(\varepsilon^{1/3}/\varepsilon)$.
- The periodic rebuilds cost $O(\varepsilon^{-1})$ and happen every $\lfloor \varepsilon^{-1/3} \rfloor$ updates.

$\square$

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Limitations of the simple allocator:

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Solution ideas:

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- Use Kuszmaul's Allocator to handle items with size $< \varepsilon^4$.
- Main difficulty is extending simple allocator to handle sizes $[\varepsilon^4, 1]$.

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- Randomized rebuilds.

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Stochastic Items: Alternating inserts of items with random sizes and deletes of random items.

Theorem

There is an allocator for stochastic items of sizes in $[\varepsilon, 2\varepsilon)$ *with worst-case expected update cost* $O(\log \varepsilon^{-1})$.

Stochastic Items — Proof Ideas

- When an item is deleted, group it together with a set of $\Theta(\log \varepsilon^{-1})$ surrounding items, and call the size of this group y.
- A random set of $\Theta(\log \varepsilon^{-1})$ items has good probability of having a subset sum which is close to y.
- Replace the deleted item and its group with a subset of a block near the end of memory.
- Compact the end of memory.

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Lower Bounds?

Theorem

There is no "resizable" allocator with expected update cost $o(\log \varepsilon^{-1})$.

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- Small items
- Stochastic items
- Few distinct types of items

Question

Is there an allocator with expected update cost $o(\varepsilon^{-1/2})$?