Scheduling Jobs with Work-*Inefficient* Parallel Solutions

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Engineer writes a serial and parallel implementation for each task and lets the *scheduler* decide which implementations to use.

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This motivates the algorithmic problem that we consider.

Defining The Serial Parallel Decision Problem

- Input: Set of n tasks (σ_i, π_i, t_i)
- σ_i = serial work, π_i = parallel work, t_i = arrival time.
- Output: serial/parallel decisions and job schedule.

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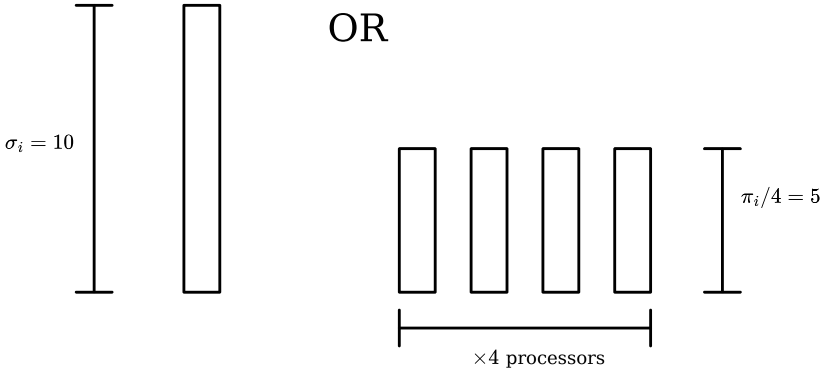
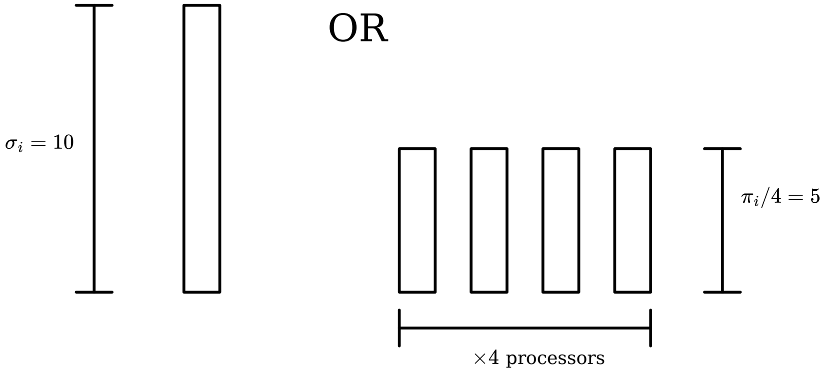


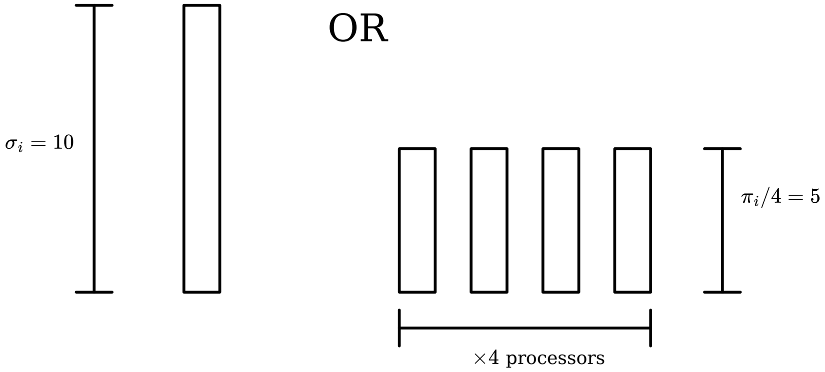
Defining The Serial Parallel Decision Problem

At each time step: allocate *p* processors to jobs.
(Serial job $\implies$ at most one processor at a time.)



Defining The Serial Parallel Decision Problem

Completion criterion: Suppose job i has work $w \in \{\pi_i, \sigma_i\}$. Let $x_i(t)$ denote the number of processors allocated to job i at time t. Job i is completed once $\int_0^T x_i(t)dt = w$.



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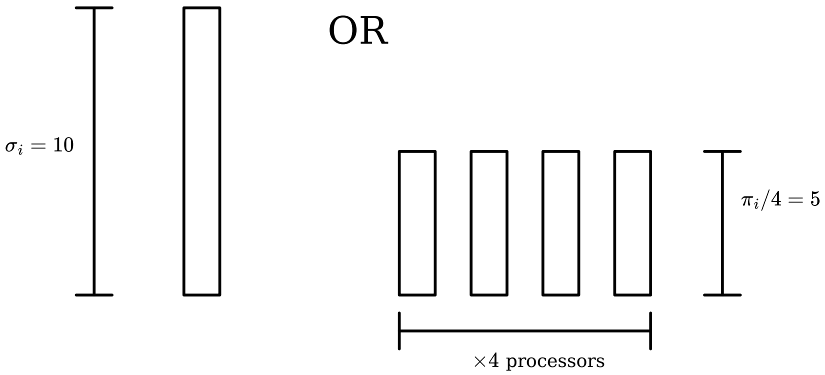
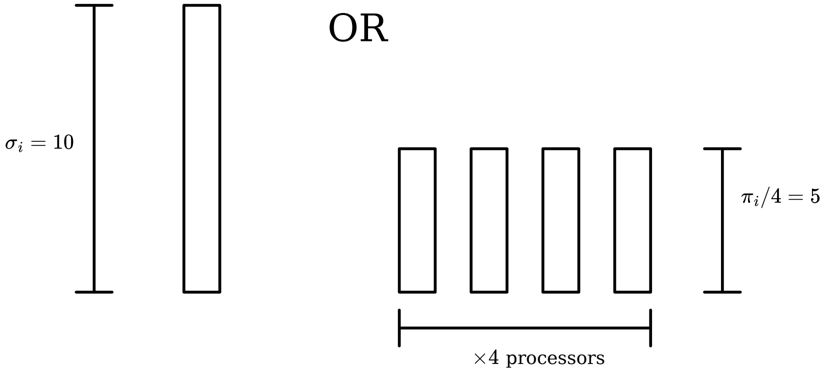
We require $\pi_i/p \leq \sigma_i \leq \pi_i$.



Defining The Serial Parallel Decision Problem

We've now described the model.
Next: discuss the scheduler's objectives.



Metric 1: Awake Time

Amount of time when there are uncompleted tasks.

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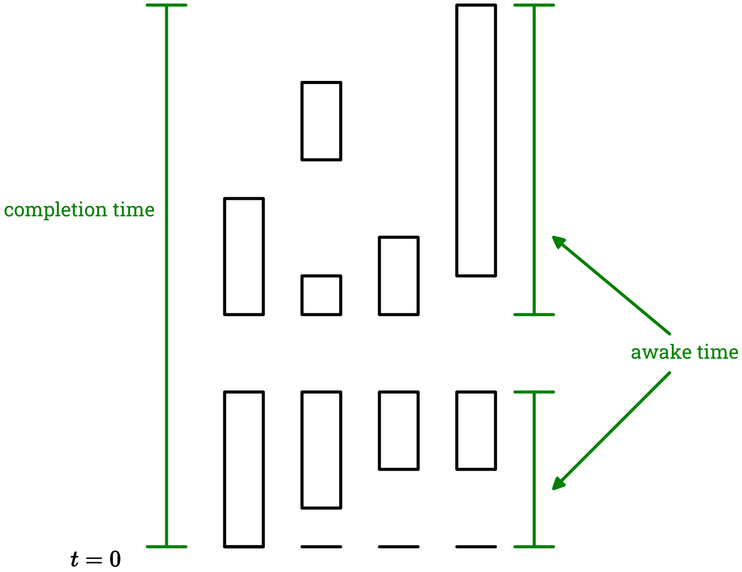

Metric 2: Mean Response Time (MRT)

Average time between receiving a task and completing it.

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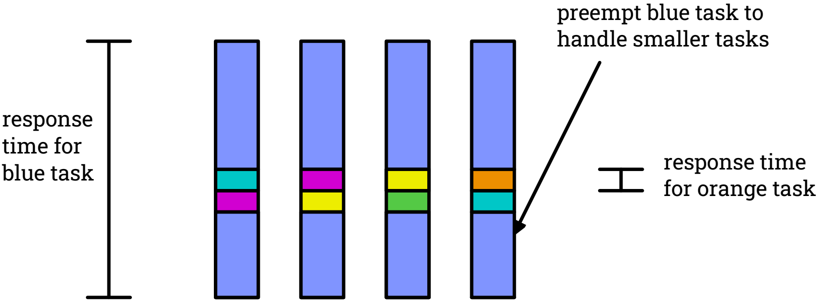

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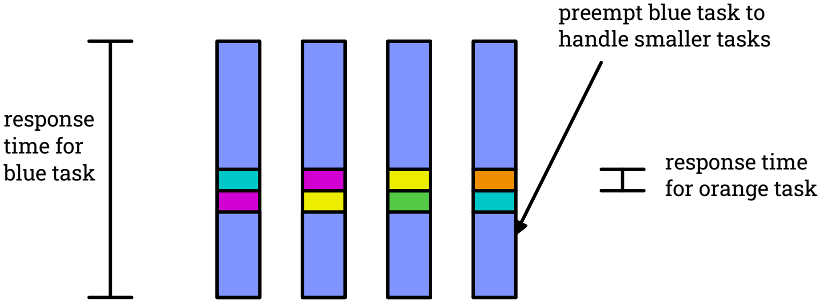

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Next: main results.

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Theorem 1

There is an $O(1)$-competitive scheduler for awake time.

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Next: Awake time specific results.

Optimizing Awake Time with Additional Restrictions

Theorem 3

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Remark 1

Any scheduler that is both decide on arrival and parallel work oblivious is not $o(\sqrt{p})$-competitive for awake time.

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Remainder of Talk:
Description and analysis of parallel work oblivious scheduler.

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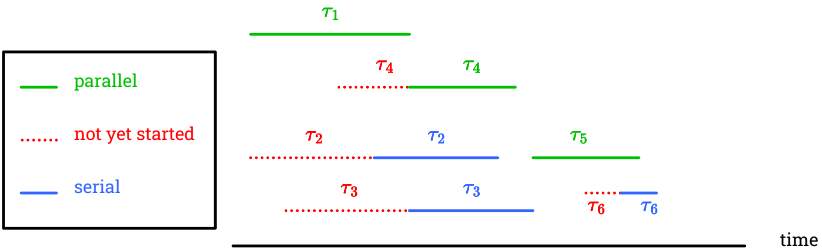

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Next: Proof outline.

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Theorem 5

PRO *is 6-competitive for awake time.*

WLOG: consider task sequences where PRO always has at least one uncompleted task present.

Proof outline:

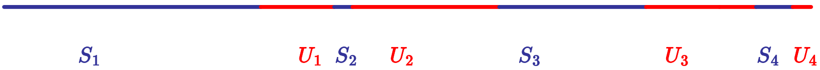
1. Show that at PRO has no idle processors at least half of the time.
2. Bound the amount of work that PRO takes.

Analysis of PRO

Saturated time step: no idle processors.

S_i saturated intervals

U_i unsaturated intervals

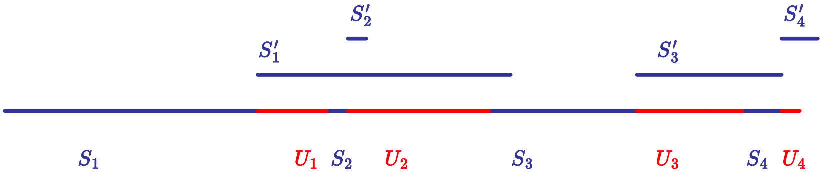


Analysis of PRO

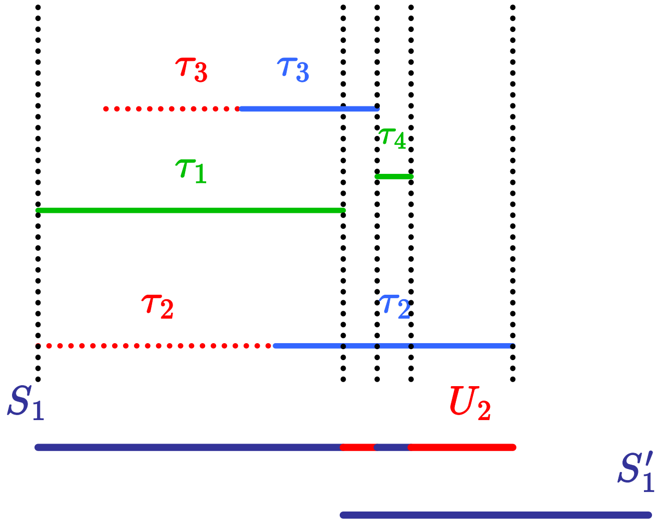
Lemma 6

PRO *is saturated at least* $1/2$ *of the time.*

Proof: Let S_i' be a copy of S_i, shifted to start at the end of S_i. We claim that $\bigcup_j U_j \subseteq \bigcup_k S_k'$.



Lemma Proof Sketch



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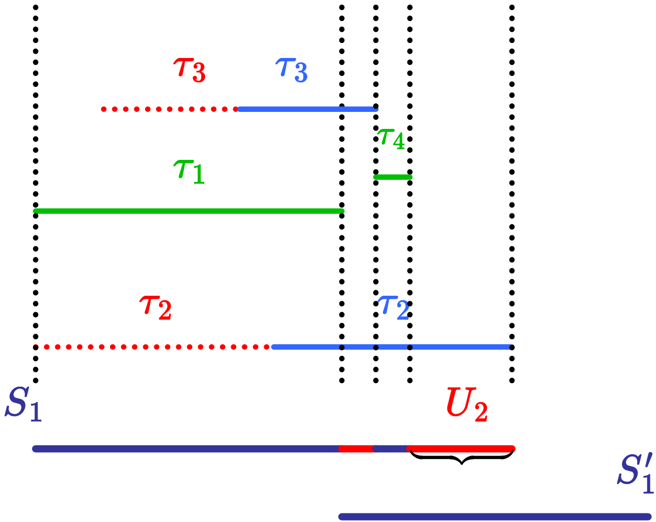

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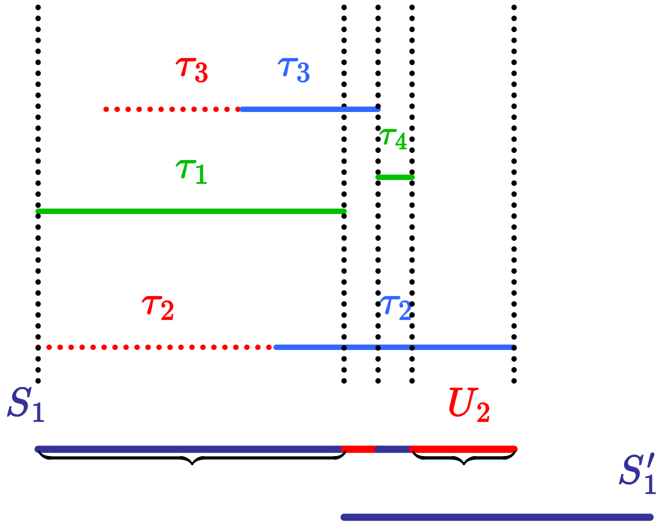

Lemma: PRO's saturated time is at most $3T_{OPT}$

T_{OPT}: optimal awake time on the tasks.

Lemma 7

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Proof idea:

Bound work on each of four (non-exclusive) categories of tasks τ.
Proof omitted due to time.

PRO Analysis: Combining the Lemmas

Theorem 8

PRO *is a 6-competitive parallel work oblivious scheduler for awake time.*

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Proof: PRO is saturated for at least $1/2$ of its time steps, and has at most $3T_{OPT}$ saturated time steps. □

Open Questions

Awake Time

model	lower bound	best algorithm
vanilla	$1.618 - O(1/p)$	2
decide on arrival	$2 - O(1/p)$	3
parallel work oblivious	$2 - O(1/p)$	6
randomized	$1.18 - O(1/p)$	2

Mean Response Time

model	lower bound	best algorithm
$O(1)$ speed augmentation	??	$O(1)$
decide on arrival	??	
parallel work oblivious with $O(1)$ speed augmentation	$\Omega(p^{1/4})$	
non-preemptive	∞	
no speed augmentation	??	

Extra slides

Decide on Arrival Scheduler

Decide on Arrival Scheduler Definition

Fix TAP $\tau_1, \tau_2, \ldots, \tau_n$.

Definition 9

C^i_{ALG}: completion time of scheduler ALG on tasks $\tau_1, \tau_2, \ldots, \tau_i$.

Scheduler BAL: When task τ_i arrives,

- If $\sigma_i + t_i \geq C^i_{\mathsf{BAL}}$ run τ_i in serial.
- Else, run τ_i in parallel.

Depiction of BAL

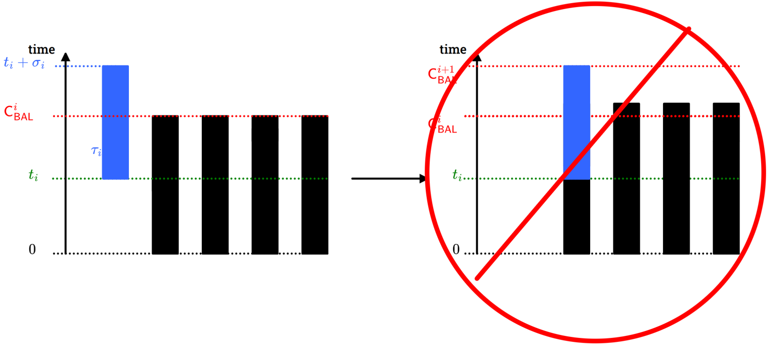

Figure: Serial job is too large: BAL chooses parallel job

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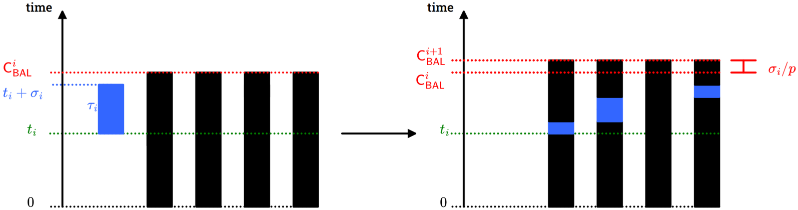

Figure: BAL chooses a serial job

Observe BAL is always "***balanced***": never has idle processors.

Key Invariant

Let OPT denote the optimal schedule of $\tau_1, \tau_2, \ldots, \tau_n$.
Important: OPT is not optimal on the first i tasks, is optimal overall.
Let K^i_{OPT} denote the work of OPT on the first i tasks.

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Lemma 10

$$\mathsf{C}^i_{\text{BAL}} \leq 2\mathsf{C}^i_{\text{OPT}} + \mathsf{K}^i_{\text{OPT}}/p.$$

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$$\mathrm{C}^i_{\mathrm{BAL}} \leq 2\mathrm{C}^i_{\mathrm{OPT}} + \mathrm{K}^i_{\mathrm{OPT}}/p.$$

Immediate corollary: BAL is 3-competitive for completion time.
(Later: extend to awake time.)

Proof of Key Invariant

Assume

$$\mathsf{C}_{\mathsf{BAL}}^{i-1} \leq 2\mathsf{C}_{\mathsf{OPT}}^{i-1} + \mathsf{K}_{\mathsf{OPT}}^{i-1}/p.$$

Case 1: BAL **runs** τ_i **in serial.**

$$\begin{aligned} \mathsf{C}_{\mathsf{BAL}}^{i} &= \mathsf{C}_{\mathsf{BAL}}^{i-1} + \sigma_i/p \\ &\leq 2\mathsf{C}_{\mathsf{OPT}}^{i-1} + (\mathsf{K}_{\mathsf{OPT}}^{i-1} + \sigma_i)/p \\ &\leq 2\mathsf{C}_{\mathsf{OPT}}^{i} + \mathsf{K}_{\mathsf{OPT}}^{i}/p. \end{aligned}$$

Proof of Key Invariant

Assume

$$C_{\mathsf{BAL}}^{i-1} \le 2C_{\mathsf{OPT}}^{i-1} + K_{\mathsf{OPT}}^{i-1}/p.$$

Case 2: BAL **and** OPT **both run** τ_i **in parallel.**

$$\begin{aligned} C_{\mathsf{BAL}}^{i} &= C_{\mathsf{BAL}}^{i-1} + \pi_i/p \\ &\le 2C_{\mathsf{OPT}}^{i-1} + (K_{\mathsf{OPT}}^{i-1} + \pi_i)/p \\ &\le 2C_{\mathsf{OPT}}^{i} + K_{\mathsf{OPT}}^{i}/p. \end{aligned}$$

Proof of Key Invariant

Assume

$$C_{BAL}^{i-1} \le 2C_{OPT}^{i-1} + K_{OPT}^{i-1}/p.$$

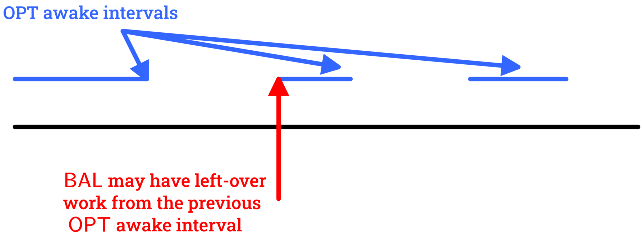
Case 3: BAL **runs** τ_i **in parallel,** OPT **runs** τ_i **in serial.**
τ_i was too large for BAL to run in serial, but OPT ran τ_i in serial:

$$C_{OPT}^{i} \ge \sigma_i + t_i \ge C_{BAL}^{i-1}.$$

Thus,

$$\begin{aligned} C_{BAL}^{i} &= C_{BAL}^{i-1} + \pi_i/p \\ &\le C_{OPT}^{i} + \sigma_i \\ &\le 2C_{OPT}^{i}. \end{aligned}$$

Extending To Awake Time



Solution: if BAL starts an awake interval with more work BAL wont get further behind on this extra work.

Extending to Awake Time

Lemma 11

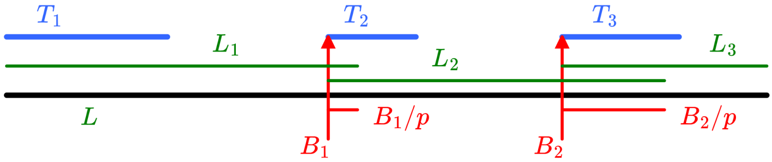
If BAL *starts (balanced) with* B *extra work and then handles the same TAP as* OPT *then*

$$\mathrm{C}_{\mathrm{BAL}} \leq 3\mathrm{C}_{\mathrm{OPT}} + B/p.$$

Extending to Awake Time

Theorem 12

BAL *is a 3-competitive decide on arrival scheduler for awake time.*



T_1, T_2, T_3: OPT completion times
L_1, L_2, L_3: BAL completion times
L: BAL total completion time
B_1, B_2: extra work

$L_1 \leq 3T_1 + 0$
$L_2 \leq 3T_2 + B_1/p$
$L_3 \leq 3T_3 + B_2/p$

$$L = L_1 - B_1/p + L_2 - B_2/p + L_3 \leq 3(T_1 + T_2 + T_3)$$

Parallel Work Oblivious Scheduler – Analysis

Analysis of PRO

Saturated time step: no idle processors.

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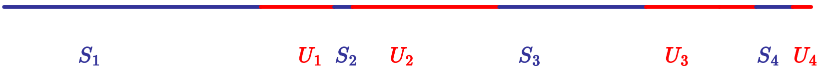
U_i unsaturated intervals



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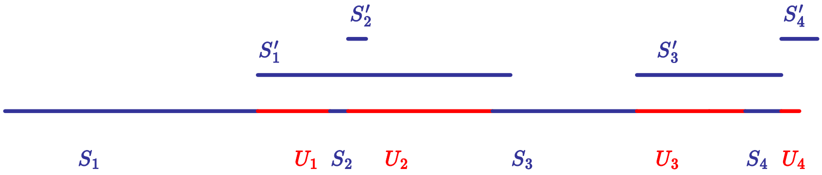
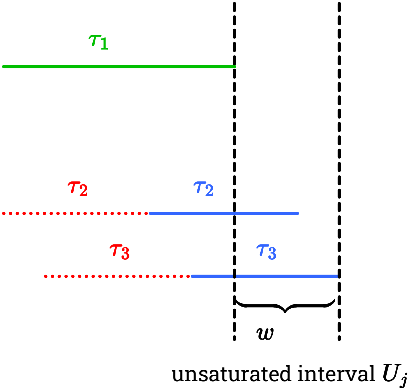


Lemma: PRO is saturated at least 1/2 the time

Claim 1

Let w be maximum over tasks i present at the start of U_j of the serial work remaining on task i. Then, $|U_j| \leq w$.

Proof:

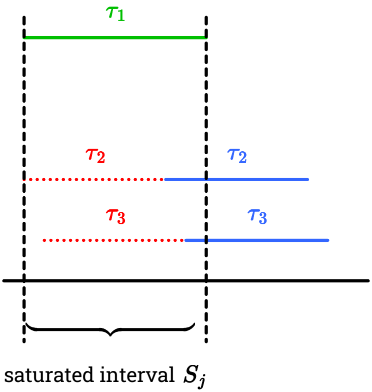


Lemma: PRO is saturated at least 1/2 the time

Claim 2 (2)

Suppose task i is started in serial during saturated interval S_j. Then, $|S_j| \geq \sigma_i$.

Proof:



Lemma: PRO is saturated at least $1/2$ the time

Claim 3 (3)

Suppose that task i is started in serial at time t and runs during an unsaturated interval $U_j = [a, b]$. Then task i is allocated a processor at each step in $[t, a]$.

Proof: If serial task i gets work stolen from it at some time t, then PRO must have p serial tasks with at least as much remaining work as task i at time t. Then, PRO will remain saturated (at least) until task i is finished.

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Corollary 14

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- Then, Claim 1 $\implies$ $|U_j| \leq \sigma_i - (a - t)$.
- So $U_j \subseteq [a, a + \sigma_i - (a - t)] = [a, t + \sigma_i] \subseteq [t, t + \sigma_i]$.

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- So $U_j \subseteq [t, t + |S_k|]$. □

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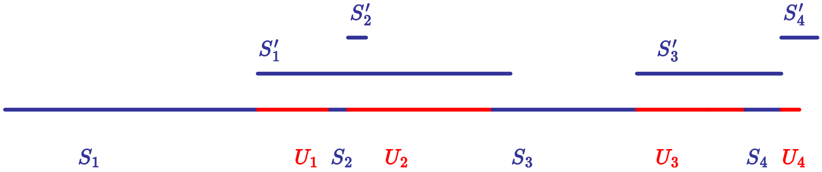
We have shown $\bigcup_j U_j \subseteq \bigcup_k S'_k$, which gives:

Lemma 15

PRO *is saturated at least* 1/2 *of the time.*



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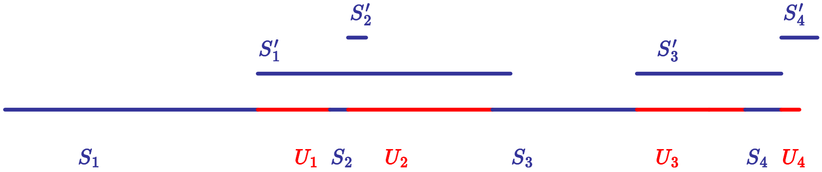

Next: bound saturated time by analyzing PRO's work.

Lemma: PRO's saturated time is at most $3T_{OPT}$

T_{OPT}: optimal awake time on the tasks.

Lemma 16

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Proof idea:

Bound work on each of four (non-exclusive) categories of tasks τ:

1. PRO runs τ is serial.
2. PRO runs τ in parallel starting after OPT finishes τ.
3. PRO runs τ in parallel completely during times when OPT has uncompleted tasks.

Lemma: PRO's saturated time is at most $3T_{OPT}$

T_{OPT}: optimal awake time on the tasks.

Lemma 16

The amount of time that PRO *is saturated is at most* $3T_{OPT}$.

Proof idea:

Bound work on each of four (non-exclusive) categories of tasks τ:

1. PRO runs τ is serial.
2. PRO runs τ in parallel starting after OPT finishes τ.
3. PRO runs τ in parallel completely during times when OPT has uncompleted tasks.
4. PRO runs τ in parallel starting before OPT finishes τ, but PRO's execution of τ overlaps with a time when OPT has no uncompleted tasks.

PRO Analysis — Type 1 and 2 Tasks

Type 1: PRO runs τ is serial.
Type 2: PRO runs τ in parallel starting after OPT finishes τ.

Claim 4 (1,2)

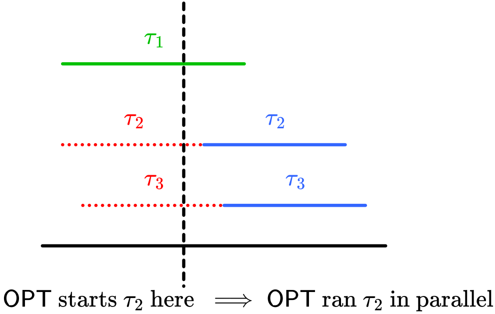
PRO spends at most $p\mathsf{T}_{\mathsf{OPT}}$ work on tasks of types (1) and (2).

PRO Analysis — Type 1 and 2 Tasks

Type 1: PRO runs τ is serial.

Type 2: PRO runs τ in parallel starting after OPT finishes τ.

Proof: If τ_i is a type (2) task then OPT finishes τ_i faster than σ_i, or else PRO would have started τ_i in serial. Thus, OPT must run type (2) tasks in parallel.



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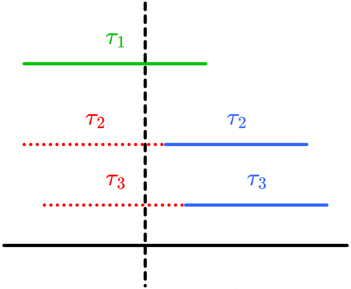


OPT starts τ_2 here $\implies$ OPT ran τ_2 in parallel

Thus, the total work performed by OPT is at least the sum of π_i for type (2) tasks and σ_i for type (1) tasks.

PRO Analysis — Type 3 Tasks

Type 3: PRO runs τ in parallel completely during times when OPT has uncompleted tasks.

Claim 4 (3)

PRO spends at most $p\mathrm{T}_{\mathrm{OPT}}$ work on tasks of types (3).

Proof: Clear.

PRO Analysis — Type 4 Tasks

Type 4: PRO runs τ in parallel starting before OPT finishes τ, but PRO's execution of τ overlaps with a time when OPT has no uncompleted tasks.

Claim 5 (4)

PRO spends at most $p\mathsf{T}_{\mathsf{OPT}}$ work on tasks of types (4).

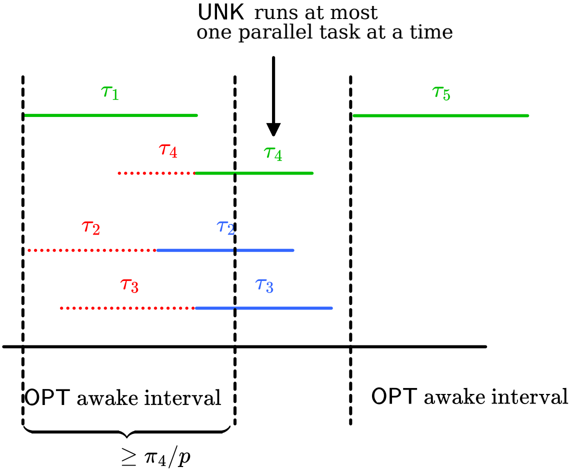
PRO Analysis — Type 4 Tasks

Type 4: PRO runs τ in parallel starting before OPT finishes τ, but PRO's execution of τ overlaps with a time when OPT has no uncompleted tasks.

Proof: For each OPT awake interval I there is at most one type (4) task that starts during I in parallel and runs past the end of I. The length of I is at least π_i/p for this type (4) task.

PRO Analysis — Type 4 Tasks

Type 4: PRO runs τ in parallel starting before OPT finishes τ, but PRO's execution of τ overlaps with a time when OPT has no uncompleted tasks.



PRO Analysis: Combining the Lemmas

Theorem 17

PRO *is a 6-competitive parallel work oblivious scheduler for awake time.*

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Proof: PRO is saturated for at least $1/2$ of its time steps, and has at most $3T_{OPT}$ saturated time steps. □