A Nearly Quadratic Improvement for Memory Reallocation




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SPAA' 2024

The Memory Reallocation Problem

memory:

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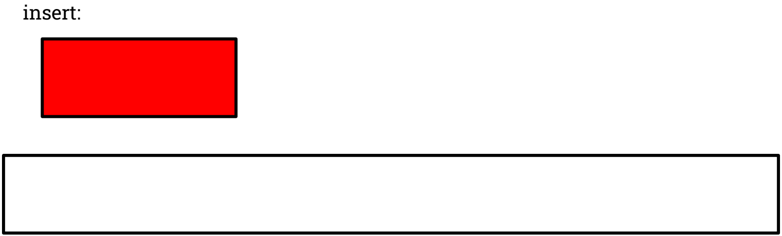
- **Input**: a sequence of item inserts and deletes.



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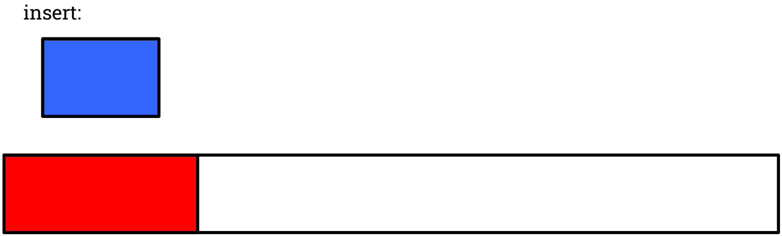

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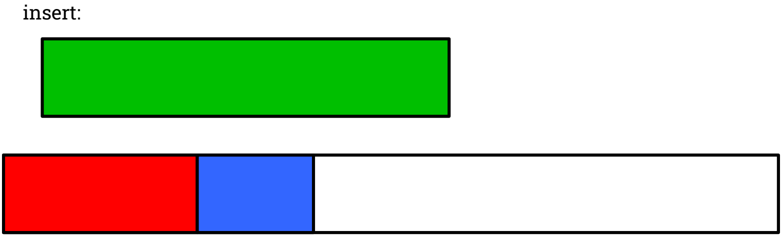

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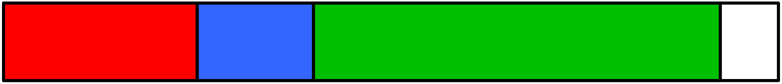

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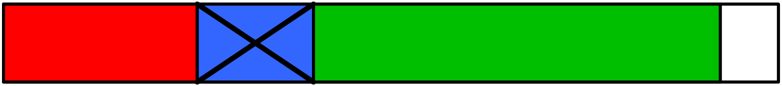

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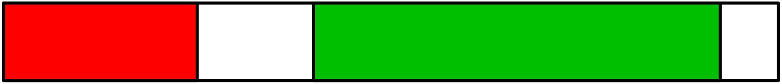

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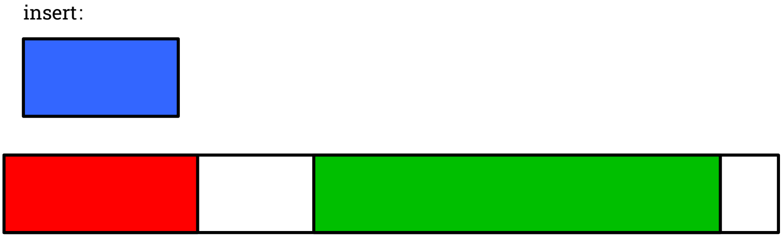

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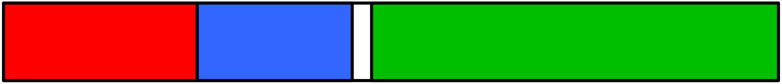

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Goal: Minimize update cost while handling load factor $1 - \varepsilon$.

Background

Proposition (Folklore Algorithm)

There is an allocator with update cost $O(\varepsilon^{-1})$.

Proof.

$\square$

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- Delete: do nothing.
- Suppose an item of size k must be inserted.
- By averaging, can show there is a length $2k\varepsilon^{-1}$ interval with k free space.
- Re-arrange this interval and place the inserted item in it.

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Proposition (Folklore Algorithm)

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Theorem (Kuszmaul FOCS'23)

If all items have size at most $O(\varepsilon^4)$ then there is an allocator with expected update cost $O(\log \varepsilon^{-1})$.

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Main Result

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Up next:
Prove a simpler version of this theorem to illustrate some ideas.

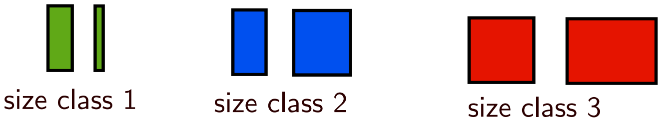
Theorem

There is an allocator for items with sizes in $[\varepsilon, 2\varepsilon]$ *with average update cost* $O(\varepsilon^{-2/3})$.

Allocator Description

Partition $[\varepsilon, 2\varepsilon)$ into $\lceil \varepsilon^{-1/3} \rceil$ **size classes**.
i-th size class:

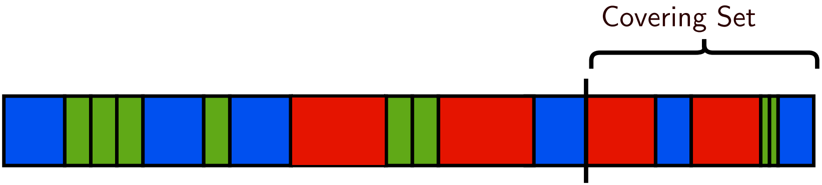
$$[\varepsilon + (i-1)\varepsilon^{4/3}, \varepsilon + i\varepsilon^{4/3}).$$



Allocator Description

Covering Set: Suffix of memory.

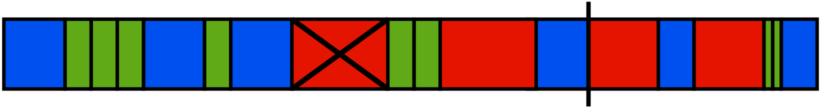
Every $\lfloor \varepsilon^{-1/3} \rfloor$ updates the allocator performs an expensive **rebuild operation** where it rearranges all of memory to place the smallest $\lfloor \varepsilon^{-1/3} \rfloor$ items of size class *i* in the covering set (or all items of size class *i* if there are fewer than $\varepsilon^{-1/3}$ items of size class *i*).



Allocator Description

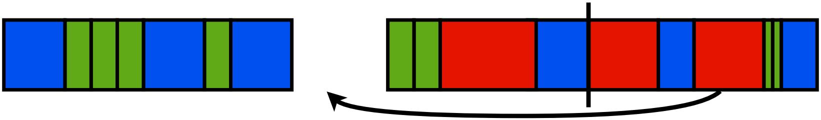
Item Deletes:

An item is deleted.



Allocator Description

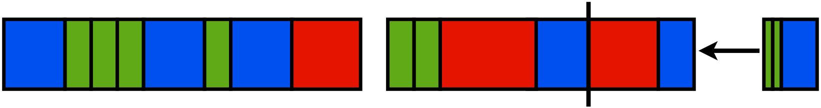
Item Deletes:
Replace item with smaller item of same size class from the covering set.



Allocator Description

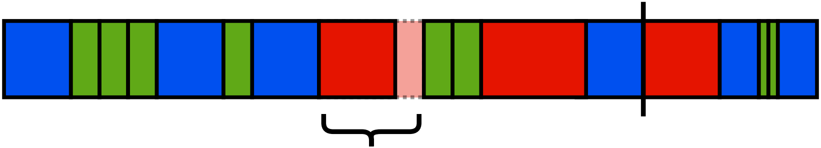
Item Deletes:

Compact covering set.



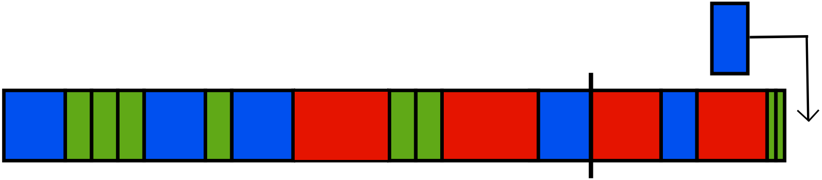
Allocator Description

Item Deletes:
Logically inflate item size.



Allocator Description

Item Inserts: Add inserted items to the covering set.
Place them after the final item in memory.
(**Why is there room for this item?**)



Allocator Analysis

Lemma

The allocator is well defined and produces a valid allocation.

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The allocator achieves amortized update cost $O(\varepsilon^{-2/3})$.

Main Open Question

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- Stochastic items.
- Items with sizes that are powers of two.
- Constant number of item sizes.

Extra Slides

Allocator Correctness

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Proof.

Periodic rebuilds prevent gaps from building up too much: we introduce up to $\varepsilon^{4/3}$ gap per delete, and rebuild after $\lfloor \varepsilon^{-1/3} \rfloor$ updates. □

Allocator Performance

Lemma

The allocator achieves amortized update cost $O(\varepsilon^{-2/3})$.

Proof.

- The covering set consists of at most $O(\varepsilon^{-2/3})$ items, and so has total size at most $O(\varepsilon^{1/3})$.
- Compacting the covering set on each delete thus costs $O(\varepsilon^{1/3}/\varepsilon)$.
- The periodic rebuilds cost $O(\varepsilon^{-1})$ and happen every $\lfloor \varepsilon^{-1/3} \rfloor$ updates.

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Limitations of the simple allocator:

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Solution ideas:

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Solution ideas:

- Use Kuszmaul's Allocator to handle items with size $< \varepsilon^4$.
- Main difficulty is extending simple allocator to handle sizes $[\varepsilon^4, 1]$.

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Solutions:

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- Number of items allowed in each size class is inversely proportional to the item size.
- Randomized rebuilds.

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Definition

Stochastic Items: Alternating inserts of items with random sizes and deletes of random items.

Theorem

There is an allocator for stochastic items of sizes in $[\varepsilon, 2\varepsilon)$ *with worst-case expected update cost* $O(\log \varepsilon^{-1})$.

Stochastic Items — Proof Ideas

- When an item is deleted, group it together with a set of $\Theta(\log \varepsilon^{-1})$ surrounding items, and call the size of this group y.
- A random set of $\Theta(\log \varepsilon^{-1})$ items has good probability of having a subset sum which is close to y.
- Replace the deleted item and its group with a subset of a block near the end of memory.
- Compact the end of memory.