Complexity of Art Gallery Variants

Nathan Sheffield, Alek Westover

MIT

Geometric Computing

The Art Gallery Problem

Input: Polygon P, number of guards g.
Output: Is there a placement of g guards that can see all of P?

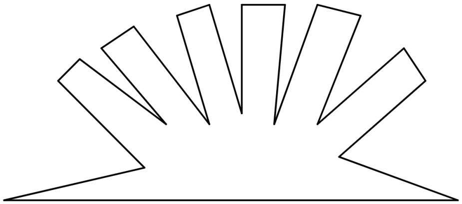

Figure: An Art Gallery

Polygon representation: P is represented as a list of n vertices which are pairs of B bit binary numbers in $\left\{ i \cdot 2^{-B} \mid i \in [2^B] \right\}$.

Can't Always Place Guards at Vertices

Theorem (Chvatal)

$\lfloor n/3 \rfloor$ vertex guards always suffice.

We can efficiently compute a set of $\lfloor n/3 \rfloor$ guards that suffice by triangulating P and 3-coloring the resulting graph.

Can't Always Place Guards at Vertices

Theorem (Chvatal)

$\lfloor n/3 \rfloor$ *vertex guards always suffice.*

But, if we want to *minimize* the number of guards, it's useful to place guards off of vertices.

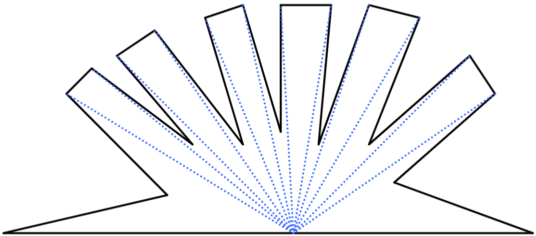

Figure: One guard suffices, but need many vertex guards.

Can't Always Place Guards at Vertices

Theorem (Chvatal)

$\lfloor n/3 \rfloor$ *vertex guards always suffice.*

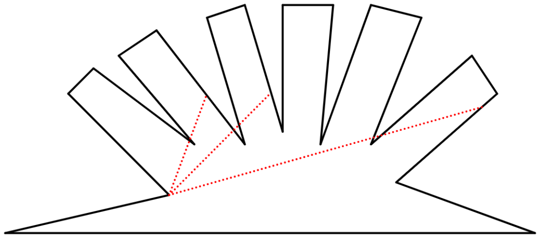

Figure: One guard suffices, but need many vertex guards.

Can't Always Place Guards at Vertices

Theorem (Chvatal)

$\lfloor n/3 \rfloor$ *vertex guards always suffice.*

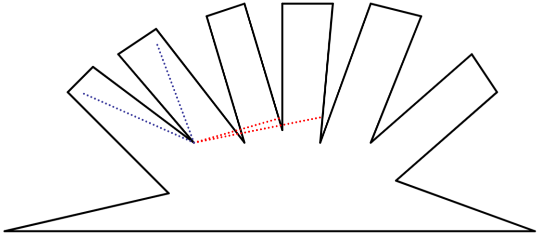

Figure: One guard suffices, but need many vertex guards.

Can't Always Place Guards at Vertices

Theorem (Chvatal)

$\lfloor n/3 \rfloor$ *vertex guards always suffice.*

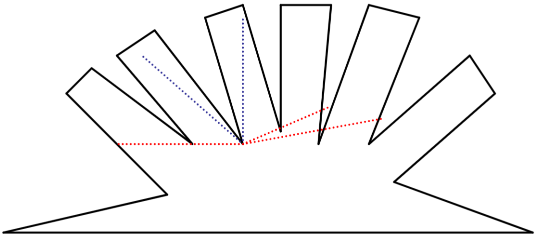

Figure: One guard suffices, but need many vertex guards.

Complexity of Art Gallery

Question: Is Art Gallery in NP?
Answer: Probably not.

Definition ($\exists\mathbb{R}$ formula)

$\exists\mathbb{R}$ is the set of all true formulas

$$\exists X_1, X_2, \ldots, X_n \in \mathbb{R} \mid \Phi(X_1, \ldots, X_n)$$

where Φ is a well-formed sentence involving variables $X_1, \ldots, X_n$ and symbols $\vee, \wedge, \neg, 0, 1, +, -, \cdot, (,), =, <, \leq$.

Example:

$$\exists(x, y) \mid x > 0 \wedge y > 0 \wedge x + y < 1.$$

Complexity of Art Gallery

Question: Is Art Gallery in NP?
Answer: Probably not.

Theorem (Abrahamsen, Adamaszek, Miltzow STOC'18)

Art Gallery is $\exists\mathbb{R}$-Complete.

Conjecture

NP $\subsetneq \exists\mathbb{R}$.

Why is $\exists\mathbb{R}$ potentially larger than NP?
Problems in NP have short certificates, but the variables we quantify over can be real, seems hard to write down $X_1, \ldots, X_n$ in a way that makes checking $\phi(X_1, \ldots, X_n)$ easy.

Art Gallery Variants

- **Line Guard**: is it possible to guard an art gallery with g line segments?
  Requirement: every point in the art gallery can be seen by some point on some line.
- **Shape Guard**: is it possible to guard an art gallery with g (fixed) shapes (e.g., radius ε circles)?
  Requirement: every point in the art gallery can be seen by some point in some shape.
- **3D Line Guard**: Line guard, but in 3D.
- **Promise Point Guard**: Distinguish between two cases:
  1. P can be guarded by g point guards.
  2. P can't be guarded by g ε-radius circle guards.

Results

Theorem

Line Guard is in $\exists\mathbb{R}$.

Theorem

Promise Point Guard is in NP.

Theorem

3D Line Guard is $\exists\mathbb{R}$*-hard.*

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

Try 1:

$\exists k$ line segments such that $\forall (x, y) \in P$,
$\exists (x', y')$ a point on one of our line segments such that
(x', y') can see (x, y).

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

Try 1:

$\exists k$ line segments such that $\forall (x, y) \in P$,
$\exists (x', y')$ a point on one of our line segments such that
(x', y') can see (x, y).

Good: Checking if line segment contained in polygon is easy.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

Try 1:

$\exists k$ line segments such that $\forall (x, y) \in P$,
$\exists (x', y')$ a point on one of our line segments such that
(x', y') can see (x, y).

Good: Checking if line segment contained in polygon is easy.

Bad: $\forall$ quantifiers not allowed in $\exists\mathbb{R}$!

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

Try 1:

$\exists k$ line segments such that $\forall (x, y) \in P$,
$\exists (x', y')$ a point on one of our line segments such that
(x', y') can see (x, y).

Good: Checking if line segment contained in polygon is easy.

Bad: $\forall$ quantifiers not allowed in $\exists\mathbb{R}$!

Try 2: Maybe there is a small set of points C such that if we can see all points in C then we can also see all points in P?

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

Try 1:

$\exists k$ line segments such that $\forall (x, y) \in P$,
$\exists (x', y')$ a point on one of our line segments such that
(x', y') can see (x, y).

Good: Checking if line segment contained in polygon is easy.

Bad: $\forall$ quantifiers not allowed in $\exists\mathbb{R}$!

Try 2: Maybe there is a small set of points C such that if we can see all points in C then we can also see all points in P?

Kind of...

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

Definition

$$X = \{\text{corners of polygon}\} \cup \{\text{guard segment endpoints}\}$$

$$\mathcal{L} = \{\text{Lines joining points in } X\}$$

$$\mathcal{R} = \{\text{regions in the arrangement defined by } \mathcal{L}\}$$

$$C = \{\text{Centroids of regions in } R\}$$

Claim: If the guard segments can see all points in C then they can see the entire polygon P.

Note: C depends on the guard locations. This is to be expected, because we do not believe that LineGuard $\in$ NP.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

Lemma

If guard segment ℓ can see any point inside region $R \in \mathcal{R}$ then ℓ can see all of R.

Corollary

If the guard segments can all points in C then they can see all points in P.

Corollary

LineGuard $\in \exists\mathbb{R}$.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

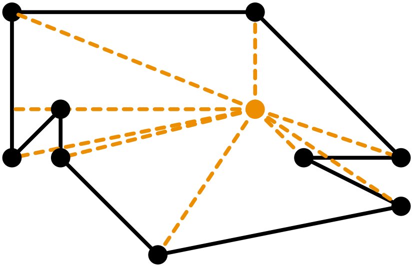

Figure: If a point $x \in X$ (i.e. a polygon vertex or a guard segment endpoint) can see any point inside region $R \in \mathcal{R}$, then that point can see the whole region.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

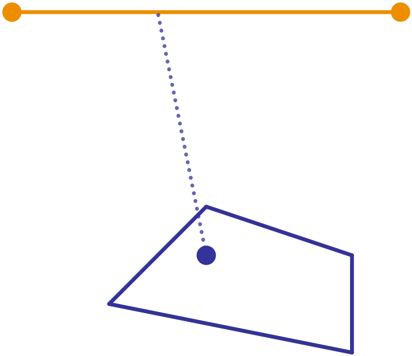

Figure: Suppose some point along a guard segment can see some point of a region R.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

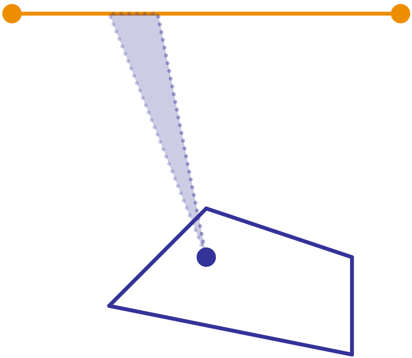

Figure: Suppose some point along a guard segment can see some point of a region R. Sweep the line of sight passing through that point.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

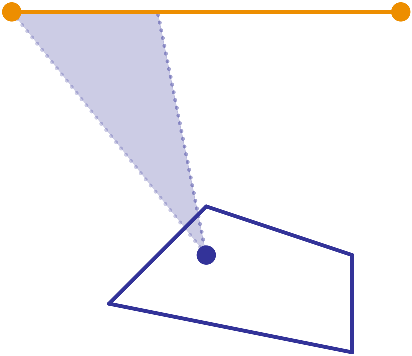

Figure: Sweep to endpoint without encountering an obstacle $\implies$ endpoint of guard segment can see a point of $R \implies$ endpoint can see all of R.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

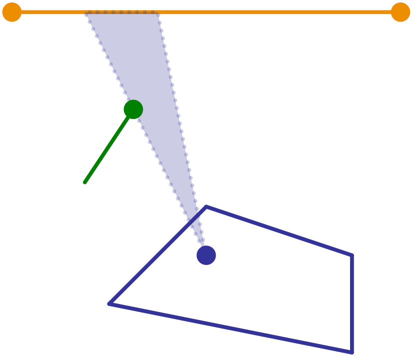

Figure: So, suppose sweep encounters an obstacle.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

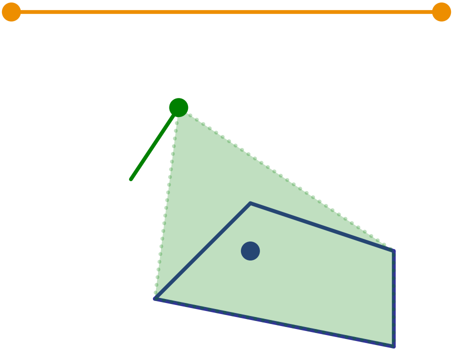

Figure: So, suppose sweep encounters an obstacle. Obstacle is a vertex of P that sees some point of R, so sees all of R.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

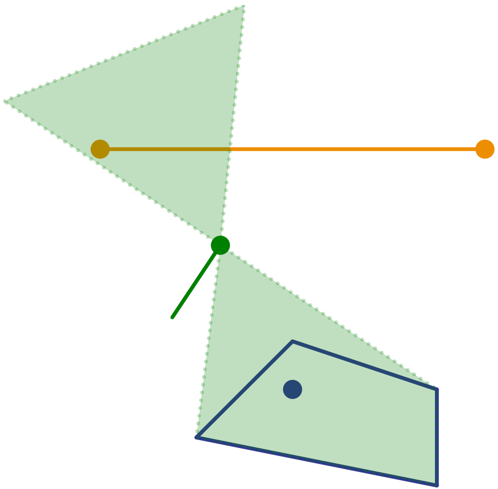

Figure: Look at the angle for which R is in the "field of view" of the obstacle, and extend backwards.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

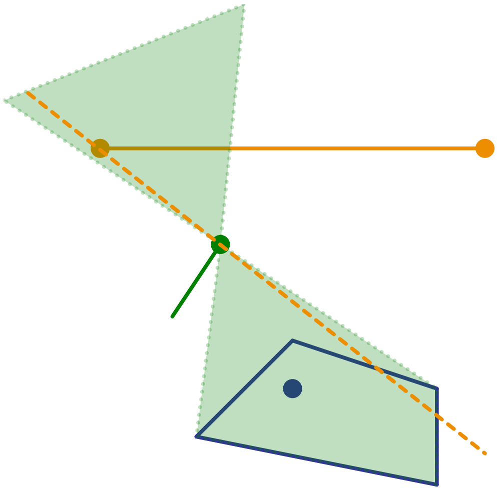

Figure: Any point from X in this "field of view" would generate a line in the arrangement splitting R, contradicting the fact that R is a region of the arrangement. So, guard segment endpoints not in field of view.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

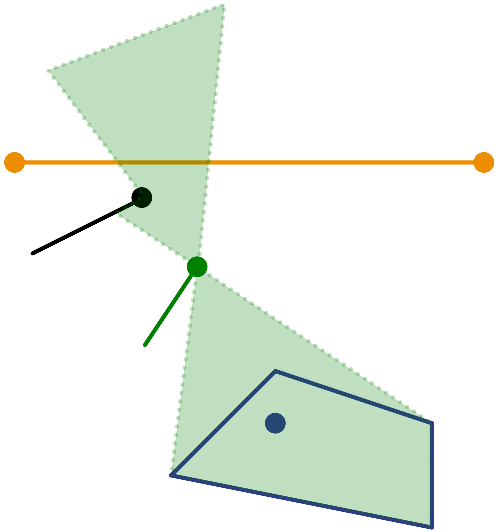

Figure: Also, no other obstacles in field of view to restrict vision from the guard segment to the green point.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

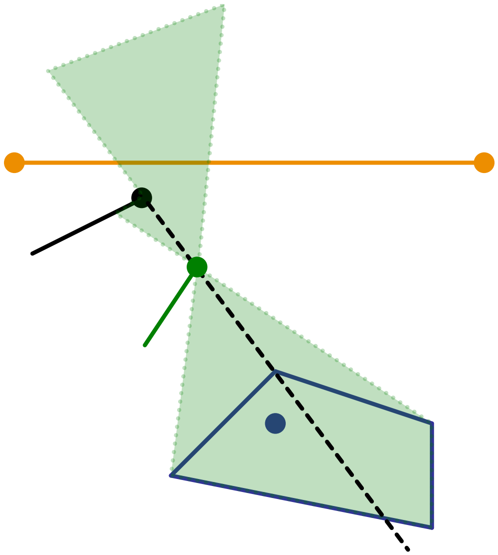

Figure: Also, no other obstacles in field of view to restrict vision from the guard segment to the green point, because they would likewise split R.

LineGuard $\in \exists\mathbb{R}$ Proof Sketch

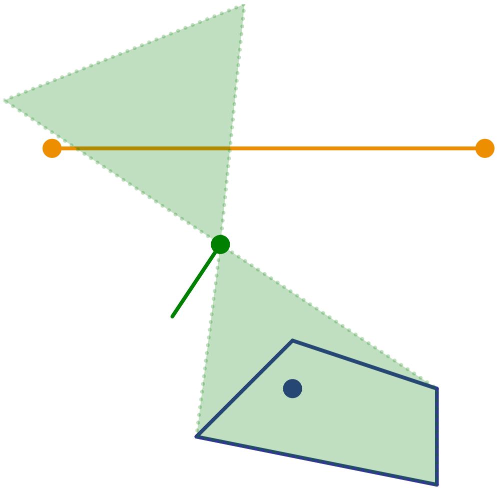

Figure: But now, for every line from the green point to a point in R, extending that line backwards must intersect the guard segment before it hits any other object. So guard segment sees all of R.